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Controllability of the Cauchy Problem for Fractional Differential Equations with Delay

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Abstract-In this short article, we have studied the controllability result of the Cauchy problem for fractional differential equation with delay in Banach spaces using the theory of an analytic semigroups. We shall confine in the Kuratowski measure of non-compactness and fixed point theorem. An example is also given to illustrate the results.

Keywords-Controllability; Fractional Differential Equations; Analytic Semigroup; Fixed Point Theorem.

AMS Subject Classifications: 34A08, 34A12, 34A60, 34K45.

1. INTRODUCTION

In this paper, we prove controllability of the Cauchy problem for fractional differential equations with delay of the form:

$$\begin{aligned} &{}^cD_t^q = Ax(t) + F(t, x_t) + Bu(t);\ t \in J := [0,T]; \\ &x(t) = \phi(t);\ t \in J_0 := [-b, 0]; \qquad \text{......(1.1)} \\ &x_t(\theta) = x(t+\theta);\ \theta \in [-b,0] \text{ and } \phi \in C([-b,0], X). \end{aligned}$$

Where $b, T > 0; D^q; q \in (0,1)$ is the Liouville-Caputo fractional derivative of order q. A is the infinitesimal generator of an analytic semigroup $L(.)$ uniformly bounded linear operator on X. $F : J \times J_0 \to X$ is a given function; the state function $x(t)$ takes values in $\Lambda = (J_0, X)$ and the control $u \in L^2(J,U)$ in a Banach space of admissible control functions with U as a Banach space. Let x_t represents the history of the state from $-\infty$ upto the present time t and x_t belongs to some abstract phase space λ. Also, $x_t : J_0 \to X$ is defined by $x_t(\theta) = x(t+\theta); \theta \in J_0$ and $\phi \in \Lambda$ where X is a Banach space with norm $\|.\|$. B is a bounded linear operator from U to X.

The theory of fractional differential equations have been proved to be remarkable tool and effective in the modelling of many situations in various fields of engineering and disciplines such as chemistry, physics, biology, control theory, image and signal processing, biophysics, blood flow phenomena, aerodynamics and so on. More details on fractional differential calculus theory are available in the monographs of Lakshmikanthan [19], Kilbas et al [18] and Miller and Ross [26].

Controllability is a very important component of many control systems, the controllability property plays an essential role in several control problems and both finite and infinite-dimensional spaces. In the past few years, the theorems about controllability of integrodifferential, differential, fractional differential systems has been studied by Chalishajar and Acharya ([8], [15], [25], [29]) and reference therein.

In ([1], [2]) Acharya and Panchal has proved the existence of the mild solutions for an impulsive fractional differential inclusions involving the Caputo derivative in Banach spaces and the controllability of an impulsive fractional differential inclusions involving the Caputo derivative using Sectorial operator in Banach spaces.

The paper is organized as follows: In section 2, we briefly present some basic notations and preliminaries. In section 3, we establish sufficient conditions for controllability of Cauchy problem for fractional differential equation with delay. Finally, an example is given to illustrate the results reported in section 4.

2. PRELIMINARIES

Throughout this paper, we consider $(X, \|.\|)$ as a Banach space, $C([a,b], X)$ denotes the space of the continuous function from $[a,b]$ to X with the norm

$$\|x\|_{[a,b]} = \max_{t\in[a,b]} \|x(t)\|$$

Assume

$C_0(x) := \{x(t); x(t) \in C([-\omega, T], X) \text{ and } x(t) \equiv 0, -\infty \le t \le 0\}$

with the norm

$$\|x\|_{C_0(x)} = \max_{t\in[0,T]} \|x(t)\|$$

2.1. *Definition* *[30]*

The Liouville-Caputo derivative of order q *for a function* $f \in C^1[0,\infty)$ *can be written as*

$${}^cD_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds, t > 0, 0 < q < 1$$

Since $A: D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $L(t)$ of uniformly bounded operator, there exists $M > 1$ such that $\| L(t) \| \le M$ for all $t \ge 0$. Moreover, $L(t)$ is continuous in the uniform operator topology for all $t \ge 0$,

$$i.e., \lim_{n \to 0} \| L(t+\eta) - L(t) \| = 0,\ \forall t \ge 0.$$

By [20], For $x \in X$, we define two operators $\{\phi(t)\}_{t\ge 0}$ and $\{\psi(t)\}_{t \ge 0}$ by

$$\Phi(t)x := \int_0^\infty \eta_q(\nu) L(t^q \nu) x d\nu$$

$$\Psi(t)x := q \int_0^\infty \nu_{\eta_q}(\nu) L(t^q \nu) x d\nu, 0 < q < 1$$

where

$$\eta_q(\nu) = \frac{1}{q} \nu^{-1-\frac{1}{q}} \rho_q(\nu^{-\frac{1}{q}}),$$

$$\rho_q(\nu) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \nu^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q),$$

$\nu \in (0,\infty)$, and η_q is a probability density functions defined on $(0,\infty)$ and satisfies $\eta_q(\nu) \ge 0$ for all $\nu \in (0,\infty)$ and

$$\int_0^\infty \eta_q(\nu) d\nu = 1, \int_0^\infty \nu_{\eta_q}(\nu) d\nu = \frac{1}{\Gamma(1+q)}.$$

Clearly, $\| \Phi(t) \| \le M, \| \Psi(t) \| \le \frac{M}{\Gamma(q)}, t \ge 0.$

2.2. Lemma *[14]*

$\Phi(t)$ and $\Psi(t)$ *are strongly continuous on* X *for* $t \ge 0$.

2.3. Lemma *[14]*

$\Phi(t)$ and $\Psi(t)$ *are norm-continuous on* X *for* $t > 0$.

2.4. Lemma

The linear operator

$$Wu = \int_0^t (t-s)^{q-1} \psi(t-s) Bu(s) ds$$

has an inverse operator W^{-1}*, which takes value in* $L^2(J,X)/KerW$ *and there exists two positive constants* M_4 *and* M_5 *such that*

$$\| B \| \le M_4 \text{ and} \| W^{-1} \| \le M_5.$$

2.5. Definition

A function $x \in C([-\omega, T], X)$ *satisfying the equation*

$$x(t) = \begin{cases} \phi(t), \quad t \in [-\omega, 0] \\ \Phi(t)\phi(0) + \int_0^t (t-s)^{q-1} \Psi(t-s) f(s, x_s) ds \\ \qquad + \int_0^t (t-s)^{q-1} \Psi(t-s) Bu(s) ds,\ t \in [0,T] \end{cases}$$

is called a mild solution of the problem(1.1) and we defined the control by

$$u(t) = W^{-1} \left[x_1 - \int_0^t (t-s)^{q-1} \Psi(t-s) f(s, x_s) ds \right](t)$$

Also,

$$\| u(t) \| = M_5 \left[\| x_1 \| + \frac{M M_1 M_2}{\Gamma q} \frac{T^q}{q} \right] = \zeta$$

2.6. Definition

The Problem (1.1) is said to be controllable on the interval J *if for every initial function* $\phi \in \Lambda$ *and* $x_1 \in X$ *there exists a control* $u \in L^2(J,X)$ *such that the mild solution* $x(.)$ *of (1.1) satisfies* $x(T) = x_1$.

2.7. Lemma *[17]*

Suppose $b \ge 0, \beta \ge 0$ *and* $a(t)$ *is a nonnegative function locally integrable on* $0 \le t \le T (T < +\infty)$, *and suppose* $x(t)$ *is nonnegative and locally integrable on* $0 \le t \le T$ *with*

$$x(t) = a(t) + b \int_0^t (t-s)^{\beta-1} [x(s) + Bu(s)] ds$$

on this interval, then we have that

$$x(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(bT(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} [a(s) + Bu(s)] \right] ds, \quad 0 \le t \le T.$$

2.8. Definition:(Kuratowski measure of noncompactness)

On each bounded subset D *in the Banach space* X*, define*

$$\mu(D) := \inf\{d > 0; D \text{ can be covered by a finite number of sets of diameter} < d\}$$

Then, $\mu(.)$ *is called the Kuratowski measure of noncompactness on* D.

Some basic properties of $\mu(.)$ are given in the following Lemma.

2.9. *Lemma* *([5], [22])*

Let X be a Banach space with norm $\|.\|$ and $A, C \subseteq X$ be bounded. Then

(1) $\mu(A)=0$ *iff A is relatively compact;*

(2) $\mu(A)=\mu(\bar{A})=\mu(c\bar{o}A)$, *where $c\bar{o}A$ is the closed convex hull of A;*

(3) $\mu(A)\le\mu(C)$ *when* $A\subseteq C$;

(4) $\mu(A+C)\le\mu(A)+\mu(C)$;

(5) $\mu(A\cup C)\le\max\{\mu(A),\mu(C)\}$;

(6) $\mu(A(0,r))=2r$, *where* $A(0,r)=\{x\in X \mid \|x\|\le r\}$ *if dim* $X=+\infty$.

2.10. *Lemma* *([16])*

Let X be a Banach space, $Q: X\to X$ be a completely continuous operator, if the set

$$\Lambda=\{x: x\in X, x=\lambda Qx, 0<\lambda<1\}$$

is bounded. Then Q has a fixed point.

2.11. *Lemma* *([16])*

Let X be a Banach space and T is an operator on X. If there exists a positive integer n such that T^n is a contractive map, i.e., there exists a constant $C(0\le C<1)$ such that

$$\|T^n x-T^n y\|\le C\|x-y\|,\ \forall x,y\in X,$$

then T^n has a unique fixed point on X and it is also the unique fixed point of T.

Before we give the main theorems, we require the following lemma.

2.12. *Lemma*

Let $a,b\ge 0, \beta\ge 0$. Suppose that $x(t)$ is nonnegative continuous function on $0\le t\le T$ with

$$x(t)\le a+b\int_0^t (t-s)^{\beta-1}\max_{0\le\tau\le s}[x(\tau)+Bu(\tau)]ds$$

on this interval. Then

$$x(t)\le a+a\sum_{n=1}^{+\infty}\frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}\frac{T^{n\beta}}{n\beta},\ 0\le t\le T$$

Proof: Write

$$z(t):=\max_{0\le s\le t}[x(s)+Bu(s)].$$

Then $z(t)$ is a non-decreasing nonnegative continuous function on $[0,T]$.

Given $0<t<T$. Then for any s, $0\le s\le t$,

$$x(s)\le a+b\int_0^s (s-r)^{\beta-1}v(r)dr$$
$$\le a+b\int_0^s r^{\beta-1}v(t-r)dr$$
$$\le a+b\int_0^t (t-s)^{\beta-1}v(s)ds$$

Hence,

$$v(s)\le a+b\int_0^t (t-s)^{\beta-1}v(s)ds$$

By Lemma 2.7, we have

$$v(t)\le a+[a+Bu]\int_0^t\left[\sum_{n=1}^{\infty}\frac{(bT(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}\right]ds, 0\le t\le T,$$

Therefore,

$$v(t)\le a+[a+Bu]\sum_{n=1}^{+\infty}\frac{(bT(\beta))^n}{\Gamma(n\beta)}\frac{T}{n\beta}^{n\beta},0\le t\le T.$$

3. MAIN RESULT:

In this section, we prove the controllability results for the system (1.1). We assume that f is not necessarily Lipschitz and A generate not only an analytic semigroup but also generate a compact semigroup, where X could be an infinite dimensional space.

3.1. *Theorem*

Let A be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator, and $f:[0,T]\times C([-\omega,0],X)\to X$ is continuous. If there are almost everywhere nonnegative measurable functions $l_1(t), l_2(t)$ on $[0,T]$ such that

$$\|f(t,\varphi)\|\le l_1(t)+l_2(t)\|\varphi\|_{[-\omega,0]}$$

for a.e. $t\in[0,T], \varphi\in C([-\omega,0],X)$ where

$$\sup_{t\in[0,T]}\int_0^t (t-s)^{q-1}l_1(s)ds<\infty,\ l_2(t)\in L^\infty([0,T]),$$

then for any $\phi\in C([-\omega,0],X)$, then problem (1.1) has at least one mild solution on $[-\omega,T]$.

Proof:

For every $\phi\in C([-\omega,0])$ we define

$$y(t):=\phi(t)(t\in[-\omega,0]),\ \ y(t):=\Phi(t)\phi(0)(t\ge 0).$$

By Lemma 2.2, we see that $y(t)\in C([-\omega,T],X)$.

Set

$$M_1:=\sup_{t\in[0,T]}\int_0^t (t-s)^{q-1}l_1(s)ds,\ M_2:=\|l_2\|_\infty,$$
$$M_3:=\max_{s\in[-\omega,T]}\|y(s)\|.$$

Let

$$x(t)=v(t)+y(t),\ t\in[-\omega,T],$$

Then it is obvious that x satisfies 2.1if and only if $x_0=0$ and for $t\in[0,T]$,

$$x(t)=\int_0^t (t-s)^{q-1}\varphi(t-s)[f(s,v_s+y_s)+Bu(s)]ds$$

We consider the operator $P:C_0(X)\to C_0(X)$ as follows:

$$(Px)(t)=\begin{cases}0, & t\in[-\omega,0]\\ \int_0^t (t-s)^{q-1}\varphi(t-s)[f(s,v_s+y_s) \\ \qquad +Bu(s)]ds, & t\in[0,T]\end{cases}$$

By using the Lebesgue dominant convergence theorem, it is easy to prove that $P:C_0(X)\to C_0(X)$ is continuous because f is continuous.

Set $B_r=\{x;x\in C_0(X),\|x\|_{C_0(x)}\},r>0.$

Next we will show that P is a compact operator on B_r.

Clearly, $\{(Px)(0):x\in B_r\}$ is compact.

For $t\in(0,T]$, let $0<\varepsilon_1<t,\ \varepsilon_2>0,\ x\in B_r$.

Thus, we obtained

$$\begin{aligned}(Px)(t)=&\\ &\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)+Bu(s)]dvds\\ &+\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)+Bu(s)]dvds\\ &+\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)+Bu(s)]dvds\\ &+\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)+Bu(s)]dvds\end{aligned}$$

Since $(\varepsilon_1{}^q\varepsilon_2)$ is compact, and the set

$$\left\{\begin{aligned}&\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v-\varepsilon_1{}^q\varepsilon_2)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}$$

is bounded, we see that the set

$$\left\{\begin{aligned}&L(\varepsilon_1{}^q\varepsilon_2)\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v-\varepsilon_1{}^q\varepsilon_2)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}$$

is relatively compact in X. Lemma 2.9(1) tells us that

$$\mu\left(\left\{\begin{aligned}&L(\varepsilon_1{}^q\varepsilon_2)\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v-\varepsilon_1{}^q\varepsilon_2)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}\right)=0$$

Moreover, it is clear that

$$\begin{aligned}&\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)+Bu(s)]dvds\\ &=L(\varepsilon_1{}^q\varepsilon_2)\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v-\varepsilon_1{}^q\varepsilon_2)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds.\end{aligned}$$

Thus, we get

$$\mu\left(\left\{\begin{aligned}&\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}\right)=0.$$

on the other hand, it is easy to see that there exists a positive constant C such that

$$\left\|\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)+Bu(s)]dvds\right\|$$

$$=C\int_0^{\varepsilon_2} q v_{\eta_q}(v)dv,\ \forall x\in B_r.$$

By Lemma 2.9(6), we have

$$\mu\left(\left\{\begin{aligned}&\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}\right)$$

$$\le 2C\int_0^{\varepsilon_2} q v_{\eta_q}(v)dv.$$

This means that,

$$\lim_{\varepsilon_1,\varepsilon_2\to 0_+}\mu\left(\left\{\begin{aligned}&\int_0^{t-\varepsilon_1}(t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}\right)=0.$$

Similarly we can prove that

$$\lim_{\varepsilon_1,\varepsilon_2\to 0_+}\mu\left(\left\{\begin{aligned}&\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}\right)=0,$$

$$\lim_{\varepsilon_1,\varepsilon_2\to 0_+}\mu\left(\left\{\begin{aligned}&\int_{t-\varepsilon_1}^{t}(t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s)\\ &\qquad +Bu(s)]dvds : x\in B_r\end{aligned}\right\}\right)=0.$$

By Lemma 2.9(4), we obtain

$$\mu\left(\left\{\begin{array}{l}\int_0^t (t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s) \\ \qquad + Bu(s)]dvds : x\in B_r\end{array}\right\}\right)$$

$$\le \mu\left(\left\{\begin{array}{l}\int_0^{t-\varepsilon_1} (t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s) \\ \qquad + Bu(s)]dvds : x\in B_r\end{array}\right\}\right)$$

$$+\mu\left(\left\{\begin{array}{l}\int_{t-\varepsilon_1}^{t} (t-s)^{q-1}\int_0^{\varepsilon_2} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s) \\ \qquad + Bu(s)]dvds : x\in B_r\end{array}\right\}\right)$$

$$+\mu\left(\left\{\begin{array}{l}\int_0^{t-\varepsilon_1} (t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s) \\ \qquad + Bu(s)]dvds : x\in B_r\end{array}\right\}\right)$$

$$+\mu\left(\left\{\begin{array}{l}\int_{t-\varepsilon_1}^{t} (t-s)^{q-1}\int_{\varepsilon_2}^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s) \\ \qquad + Bu(s)]dvds : x\in B_r\end{array}\right\}\right)$$

Letting $\varepsilon_1, \varepsilon_2 \to 0^+$, we get

$$\mu\left(\left\{\begin{array}{l}\int_0^t (t-s)^{q-1}\int_0^{\infty} q v_{\eta_q}(v)L((t-s)^q v)[f(s,v_s+y_s) \\ \qquad + Bu(s)]dvds : x\in B_r\end{array}\right\}\right) := 0$$

Consequently, we see that $\{(Px)(t) : x\in B_r\}$ is relatively compact in X for all $t\in[0,T]$.

Clearly, for $t\in[0,T)$,

$$\|(Px)(t)-(Px)(0)\|$$

$$\le \frac{M}{\Gamma q}\int_0^t (t-s)^{q-1}\|f(s,v_s+y_s)+Bu(s)\|ds.$$

Thus, for $0<t_1<t_2\le T$, we obtain,

$$\|(Px)(t)-(Px)(0)\|$$

$$= \left\|\int_0^{t_2}(t_2-s)^{q-1}\psi(t_2-s)[f(s,v_s+y_s)+Bu(s)]ds\right.$$

$$\left.-\int_0^{t_1}(t_1-s)^{q-1}\psi(t_1-s)[f(s,v_s+y_s)+Bu(s)]ds\right\|$$

$$\le \int_0^{t_2}(t_2-s)^{q-1}\|\psi(t_2-s)-\psi(t_1-s)\|\,\|[f(s,v_s+y_s)+Bu(s)]\|ds$$

$$+\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}\|\psi(t_1-s)\|\,\|[f(s,v_s+y_s)+Bu(s)]\|ds$$

$$+\int_{t_1}^{t_2}[(t_2-s)^{q-1}\|\psi(t_2-s)\|\,\|[f(s,v_s+y_s)+Bu(s)]\|ds.$$

This, together with Lemma 2.3, implies that $P(B_r)$ is equicontinuous on $[0,T]$. Obviously $P(B_r)$ is bounded in $C_0(X)$. By the Arzela- Ascoli theorem, we know that P is a compact operator. Hence, P is completely continuous in $C_0(X)$.

Set $\Lambda := \{l; x\in C_0(X), l=\lambda Px, 0<\lambda<1\}$. Take $l\in\Lambda$. Then for each $t\in[0,T]$,

$$l(t)=\lambda\int_0^t (t-s)^{q-1}\psi(t-s)f(s,v_s+y_s)ds.$$

Thus,

$$\|l(t)\|\le \frac{MM_1}{\Gamma q}+\frac{M}{\Gamma q}\int_0^t (t-s)^{q-1}M_2(t-s)\Big\{\Big[\|v_s\|_{[-\omega,0]}+\|y_s\|_{[-\omega,0]}\Big] + \|B\|\|u(s)\|\Big\}ds$$

$$\le \frac{MM_1}{\Gamma q}+\frac{MM_2M_3}{\Gamma q}\frac{T^q}{\Gamma q}+\frac{MM_2}{\Gamma q}\int_0^t (t-s)^{q-1}\max_{0\le T\le s}\|x(\lambda)\|ds$$

$$+\frac{MM_2M_3}{\Gamma q}\frac{T^q}{q}M_4\zeta$$

Suppose,

$$C_1=\frac{MM_1}{\Gamma q}+\frac{MM_2M_3}{\Gamma q}\frac{T^q}{\Gamma q}+\frac{MM_2M_3}{\Gamma q}\frac{T^q}{q}M_4\zeta \quad C_2=\frac{MM_2}{\Gamma q}$$

Then

$$\|l(t)\|\le C_1+C_2\int_0^t (t-s)^{q-1}\max_{0\le T\le s}\|x(\lambda)\|ds.$$

By Lemma 2.12, we have

$$\|l(t)\|\le C_1+C_2\sum_{n=1}^{+\infty}\frac{(C_2T(\beta))^n}{\Gamma n\beta}\frac{T^{n\beta}}{n\beta}<\infty,\ 0\le t\le T.$$

Therefore, the set Λ is bounded.

By Lemma 2.10, we see that P has a fixed point $l(t)$. Thus, $x(t)=v(t)+y(t)$ is the mild solution of the problem (1.1) and $x(T)=x_1$ which implies that the system (1.1) is controllable on $[0,T]$. This completes the proof of the theorem.

4. EXAMPLE:

Consider the following problem

$$\begin{cases} {}^{C}D_t{}^{q}=Au(t)+f(t,u_t),\ t\in[0,T],\\ u(t)=\phi(t),\ t\in[-\omega,0]\end{cases}$$

Where X is a Banach space, $q\in(0,1), T,\omega>0$ are constants, A is the infinitesimal generator of an analytical semigroup of uniformly bounded linear operator on a Banach space X,

$$f(t,\phi)=c_1(t)x_0+c_2(t)\int_{-\omega}^{0}\phi(s)ds,$$

$x_0\in X$ is a fixed element, $c_i(t)\ (i=1,2)$ are continuous functions on $[0,T]$, and $\phi\in C([-\omega,0],X)$. So the problem has a unique mild solution.

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